Comparison of Two Numerical Methods for the Time-Wise Integration of Transient Structural Response

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Introduction

THIS Note compares the relative efficiency of two numerical methods for the time-wise integration of the transient response of a thin, elastic shell of revolution subjected to arbitrary asymmetric loads. The two methods are 1) the numerically stable, implicit Houbolt backward differencing scheme, and 2) the explicit central differencing scheme. The efficiency of these two methods was compared in Ref. 1 using actual computer times. Their relative efficiency is determined here by counting the number of multiplications and divisions required to generate a solution at each time step. Although this procedure is not infallible, it does provide an estimate of the potential computational efficiency of a method.[2]

Equations of Motion

The set of partial differential equations governing the motion of a thin, elastic shell of revolution can be expressed as a function of a meridional coordinate s, a circumferential coordinate θ, and the time variable t. When the analysis is linear, the θ dependence can be eliminated by expressing the dependent variables in a sine or cosine series with the argument $n\theta$. The resulting sets of equations can be given in the form[1,3,4]

$$E^{(n)}z^{(n)\prime\prime} + F^{(n)}z^{(n)\prime} + G^{(n)}z^{(n)} = e^{(n)} + D\ddot{z}^{(n)} \quad n = 0,1,2,\ldots \tag{1}$$

where $E^{(n)}$,$F^{(n)}$ and $G^{(n)}$ are 4×4 matrices defined in Refs. 1 and 4, $e^{(n)}$ is the vector of Fourier coefficients of the applied load and $z^{(n)}$ is the vector of dependent variables $u^{(n)}$,$v^{(n)}$,$w^{(n)}$, and $m_s^{(n)}$, the series coefficients of the meridional, circumferential, and normal displacements, and the meridional bending moment, respectively. Superscript primes and dots denote differentiation with respect to s and t, respectively, and

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Henceforth, the superscript n will be dropped for convenience.

Equation (1) can be converted to algebraic form by replacing the spatial derivatives with the conventional central finite difference approximations. This leads to

$$A_iz_{i+1} + B_iz_i + C_iz_{i-1} = g_i + 2\Delta D\ddot{z}_i \quad i = 0,1,2\ldots I \tag{2}$$

where i denotes the meridional station, $i = 0$ is the station at the initial boundary, $i = I$ is the station at the final boundary, Δ is the meridional increment, and $A_i = 2E_i/\Delta + F_i$, $B_i = -4E_i/\Delta + 2\Delta G_i$, $C_i = 2E_i/\Delta - F_i$, $g_i = 2\Delta e_i$. The stations $i = -1$ and $i = I + 1$ are ficticious stations adjacent to each boundary of the shell. In addition to Eq. (2), there are boundary conditions on z and z' that must be satisfied. These can be given in the finite difference form

$$H_iz_{i+1} + J_iz_i + K_iz_{i-1} = l_i \quad i = 0 \text{ and } I \tag{3}$$

where H_i,J_i and K_i are 4×4 matrices and l_i is a column matrix.

Methods of Solution

The Houbolt scheme

In the Houbolt scheme $\ddot{z}_i$ is approximated by[5]

$$\ddot{z}_{i,j} = (1/\delta^2)\{2z_{i,j} - 5z_{i,j-1} + 4z_{i,j-2} - z_{i,j-3}\} \quad j > 2 \tag{4}$$

where j denotes the time step and δ is the time increment; hence $t_j = j\delta$. Substituting Eq. (4) into Eq. (2) leads to

$$A_iz_{i+1,j} + \bar{B}_iz_{i,j} + C_iz_{i,j} = \bar{g}_{i,j} \quad i = 0,1,2\ldots I \tag{5}$$

where $\bar{B}_i = B_i - (4\Delta/\delta^2)D$, and $\bar{g}_{i,j} = g_{i,j} + (2\Delta/\delta^2)D\{-5z_{i,j-1} + 4z_{i,j-2} - z_{i,j-3}\}$. Note that A_i,$\bar{B}_i$ and C_i are independent of j, provided $j > 2$. For $j = 0,1$, and 2 the scheme is not self-starting and Houbolt proposed special approximations to $\ddot{z}_{i,j}$. These approximations make $\bar{B}_i$ a function of j for the first three time steps. In the following analysis, the assumption is made that $j > 2$.

Equations (3) and (5) comprise a set of simultaneous algebraic equations that govern the motion of the shell. These equations are tridiagonal in a matrix sense, and a solution can be obtained using Potters' elimination method[6] in which the following recursive relationships are developed

$$\left.\begin{aligned} x_{i,j} &= Q_i\bar{g}_{i,j} - R_ix_{i-1,j} \\ z_{i,j} &= -P_iz_{i+1,j} + x_{i,j} \end{aligned}\right\} \quad i = 1,2\ldots I \qquad \begin{matrix}(6a)\\(6b)\end{matrix}$$

where

$$x_{0,j} = [J_0 - K_0C_0^{-1}\bar{B}_0]^{-1}l_{0,j}$$

$$P_0 = [J_0 - K_0C_0^{-1}\bar{B}_0]^{-1}[H_0 - K_0C_0^{-1}A_0]$$

$$\left.\begin{aligned} Q_i &= [\bar{B}_i - C_iP_{i-1}]^{-1} \\ P_i &= Q_iA_i \\ R_i &= Q_iC_i \end{aligned}\right\} \quad i = 1,2,\ldots I$$

and $z_{I+1,j} = [H_I - (J_I - K_IP_{I-1})P_I]^{-1}\{l_{I,j} - [J_I - K_IP_{I-1}]x_{I,j} - K_Ix_{I-1,j}\}$. A solution can be obtained at each j by computing $x_{i,j}$ for $i = 0$ to I using Eq. (6a). A back substitution using Eq. (6b) for $i = I$ to 0 determines $z_{i,j}$.

The number of multiplications and divisions (operations) required to compute a typical A_i,$\bar{B}_i$, and C_i is estimated to be approximately 250 per station when repeated quantities are stored. The number of operations required to compute a typical P_i,Q_i and R_i is 1) $C_iP_{i-1} = 64$, 2) $Q_i = [\bar{B}_i - C_iP_{i-1}]^{-1} = 93$, †3) $P_i = Q_iA_i = 64$, and 3) $R_i = Q_iC_i = 64$, for a total of 285 per station. Since P_i,Q_i, and R_i are independent of j they should be stored in core. The number of operations required to compute $x_{i,j}$ for a given $g_{i,j}$, which requires 3, is 1) $\bar{g}_{i,j} = g_{i,j} + (2\Delta D/\delta^2)\{-5z_{i,j-1} + 4z_{i,j-2} - z_{i,j-3}\} = 9$, 2) $Q_i\bar{g}_{i,j} = 16$, and 3) $R_ix_{i-1,j} = 16$, for a total of 44 per station per time step. The back substitution for $z_{i,j}$ requires 1) $P_{i,j}z_{i+1,j} = 16$. Thus, approximately 535 operations per station are initially required, and thereafter 60 operations per station per time step.

The explicit scheme

In the explicit scheme considered here $\ddot{z}_i$ is approximated by

$$\ddot{z}_{i,j} = \frac{1}{\delta^2}\{z_{i,j+1} - 2z_{i,j} + z_{i,j-1}\} \tag{7}$$

Substituting Eq. (7) into Eq. (2) gives the explicit recursion relationship

$$\begin{aligned} Dz_{i,j+1} = {} & (\delta^2/2\Delta)g_{i,j} + [2D + (\delta^2/2\Delta)B_i]z_{i,j} + \\ & (\delta^2/2\Delta)A_iz_{i+1,j} + (\delta^2/2\Delta)C_iz_{i-1,j} - Dz_{i,j-1} \\ & i = 0,1,2\ldots I \end{aligned} \tag{8}$$

The solution at $i = -1$ and $i = I + 1$ and $j + 1$ is obtained using Eq. (3). This scheme is self-starting; hence Eq. (8) holds for all j. As a consequence of the nature of D, only the

Received February 16, 1971, revision received May 13, 1971.
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† Refer to Ref. 2 for a discussion of the operations required to compute the inverse of a matrix.

first three elements of $z_{i,j+1}$ can be obtained from Eq. (8). The fourth element at $j + 1$ can be computed explicitly at each i from the fourth row of Eq. (2) using the first three elements of $z_{i,j+1}$. Since all of the 4×4 matrices in Eq. (8) are independent of j they should be stored.

The number of multiplications and divisions initially required to compute $(\delta^2/2\Delta)A_i$, $(\delta^2/2\Delta)B_i$, and $(\delta^2/2\Delta)C_i$ is essentially the same as with the Houbolt scheme. The number of operations required for the first three elements of $z_{i,j+1}$ is 1) $\delta^2/2\Delta g_{i,j} = 3$, 2) $[2D + \delta^2/2\Delta B_i]z_{i,j} = 12$, 3) $\delta^2/2\Delta A_i z_{i+1,j} = 12$, and 4) $\delta^2/2\Delta\ C_i z_{i-1,j} = 12$. The fourth element requires 10, for a total of 49 per station per time step. If account is taken of the fact that there are several zeros in A_i,B_i, and C_i, the total reduces to 44. For a cylindrical shell of uniform thickness there are many zeros in the matrices and it is more efficient to treat each multiplication individually rather than to use the general matrix multiplication procedures indicated in Eq. (8). For this case, the total reduces to 34 operations per station per time step.

Nonlinear Problems

The author has used the Houbolt scheme defined by Eq. (5) to compute the geometrically nonlinear response of arbitrarily loaded shells.[3] The nonlinearities are treated as pseudo loads and are incorporated with the load vector $g_{i,j}$. Several iterations are allowed at each time step if the latest estimated solution at j and the computed solution at j differ by more than a prescribed amount. Thus, $(60 + T)\beta$ multiplications and divisions occur at each time step where β is the number of iterations and T is the computation of the nonlinearities. If a sufficiently small time step is used, no iterations are necessary and $\beta = 1$. An alternative procedure is to include the nonlinearities in $g_{i,j}$, but use the explicit scheme defined by Eq. (8). For this case, $44 + T$ multiplications and divisions are required at each time step.

Conclusions

The Houbolt scheme requires an initial expenditure of approximately 250 multiplication and divisions per station to compute the matrices A_i,$\bar{B}_i$ and C_i plus 285 per station to compute the recursive matrices P_i,Q_i, and R_i. The transient solution requires 60 operations per station per time step. The explicit scheme requires an initial expenditure of approximately 250 operations per station to compute the A_i,B_i, and C_i matrices, and the solution takes 44 operations per station per time step for the general shell. Thus, the explicit scheme should be approximately 35% faster than the Houbolt scheme per time step.

For a uniform cylinder, the explicit scheme operations reduce to 34, and hence it should be approximately 75% faster. This conclusion is not supported by the evidence presented in Ref. 1. However, the computer times given in Ref. 1 were for an explicit method that used three variables instead of four. Communication with the senior author of Ref. 1 revealed that the details of the program based on the three variables u,v, and w differ somewhat from the four variable explicit procedure presented in Sec. 5 of Ref. 1 and in this Note. For example, the equations of motion contained the spatial derivatives instead of the differences as in Eq. (2), and the derivatives u',u'',v',v'',w',w'', and w^{iv} were efficiently computed by addition and subtraction, i.e. $\Delta^2u'' = u_{i+1} - u_i - u_i + u_{i-1}$, rather than multiplication. Using these and similar techniques, the number of required multiplications and divisions can be reduced to 15 per station per time step, excluding the effort required to generate the load terms. Thus, the ratio between the Houbolt scheme and the three variable explicit scheme used by the authors of Ref. 1 is 60/15, which is much closer to the speed ratio of 6 reported in Ref. 1. More computer time was required when the four variable method was used. The senior author of Ref. 1 estimates that approximately 30 multiplications and divisions are required when the same techniques are applied to the general shell of revolution. Thus, this other explicit scheme should be approximately 50% faster than the explicit scheme described here.

For nonlinear response, the Houbolt scheme requires $(60 + T)\beta$ multiplications and divisions, and the explicit scheme described in this note takes $44 + T$. Thus, this explicit scheme should be less than $35\beta\%$ faster than the Houbolt scheme. If the time step is small enough, $\beta = 1$.

There are several other features to consider when selecting an integration scheme, such as numerical instability and numerical damping. The Houbolt scheme is numerically stable; the explicit scheme is not. The Houbolt scheme introduces significant damping if δ becomes too large; the explicit scheme does not. Because of stability considerations a larger time increment can be used in the Houbolt scheme than in the explicit scheme, and consequently when the solution varies slowly, the Houbolt scheme can require less computation time over the total response period.

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Maximal Plastic Deformation of Semi-Infinite Rods

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THE dynamic response of structures deformed beyond the elastic limit is a function of both the magnitude and the history of the applied load $p(t)$. With any loading system, whether it is a punch press or a blast wave, there are certain constraints for example, on the available energy or the maximum pressure that can be applied. When the loading can be controlled within these constraints, it is an optimization problem to determine the $p(t)$ that results in the maximum plastic deformation. This Note examines a simple problem which shows that two common assumptions, that the loading system either applies a certain impulse or imparts a certain energy to the structure, result in distinctly different optimal load histories.

We consider a case of elasto-plastic wave propagation in a semi-infinite rod. The rod is composed of a linearly work-

Received March 22, 1971.

Index categories: Structural Dynamic Analysis; Launch Vehicle and Missile Fabrication.

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